

Gluon condensate in charmonium sum rules for the axial-vector current

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Abstract

The charmonium sum rules for the axial-vector current are considered. The three-loop perturbative corrections, operators up to 8 dimension and α_s -corrections to the lowest dimension operator are accounted. The contribution of the operators of 6 and 8 dimensions is computed in the framework of factorization hypothesis and instanton model. For the value of gluon condensate the following result is obtained: $\langle \frac{\alpha_s}{\pi} G^2 \rangle = (0.005 + 0.001 - 0.004) \text{ GeV}^4$ (for the charm quark mass $\bar{m} = 1.275 \pm 0.015 \text{ GeV}$).

1 Introduction

It is well known that the nonperturbative phenomena in QCD are of great importance. Indeed, quark and gluon condensates determine the properties of hadrons and their interactions to a considerable extent. This was stressed by Shifman, Vainshtein and Zakharov in [1], where the technique of QCD sum rules was proposed.

In particular, authors of [1] pointed out to the significance of the gluon condensate in nonperturbative QCD. First of all, gluon condensate has the lowest dimension among chirality conserving condensates and that is why it plays dominant role in the sum rules, corresponding to the processes without chirality violating. Moreover, the gluon condensate is directly related to the density of vacuum energy.

The numerical value of the gluon condensate was found in [1] from the analysis of charmonium sum rules. It was

$$\langle \frac{\alpha_s}{\pi} G_{\mu\nu}^a G_{\mu\nu}^a \rangle = 0.012 \text{ GeV}^4.$$

In this estimation the value $\Lambda_{QCD} = 100 \text{ MeV}$ was used and α_s -corrections of the first order were taken into account.

However, at the present moment Λ_{QCD} is known to be sufficiently larger, $\Lambda_{QCD} \approx 250 \text{ MeV}$, and, furthermore, α_s -corrections of the second order to the polarization operator are available. In addition, authors of [1] worked at $Q^2 = 0$, where the higher order terms in the operator expansion series are significant [2]. These facts are taken into account in the recent paper [3], where the charmonium sum rules for the vector current are considered over again with the purpose to obtain the value of $\langle \frac{\alpha_s}{\pi} G_{\mu\nu}^a G_{\mu\nu}^a \rangle$. (Of course, there were many other papers, devoted to the calculation of the gluon condensate, the short list of them can be found in [3], for review see [4]).

As the result, authors of [3] obtained $\langle \frac{\alpha_s}{\pi} G_{\mu\nu}^a G_{\mu\nu}^a \rangle = 0.009 \pm 0.007 \text{ GeV}^4$ and $\bar{m} = 1.275 \pm 0.015 \text{ GeV}$ for the c -quark mass in $\overline{\text{MS}}$ scheme. Thus one can see that the accuracy of the quark mass value is high, whereas the error in the value of the gluon condensate is comparable with its magnitude.

In the paper [5] the charmonium sum rules for the pseudoscalar current were analysed, and for the gluon condensate value the following restriction was obtained:

$$\langle \frac{\alpha_s}{\pi} G_{\mu\nu}^a G_{\mu\nu}^a \rangle \lesssim 0.008 \text{ GeV}^4.$$

In the present paper we try to obtain the gluon condensate by considering other independent channel in charmonium, namely, we analyze the axial-vector current.

It should be mentioned that the values of gluon condensate and c -quark mass are interdependent in the charmonium sum rule, i.e. the variation of one of them results in the changing of another. That is why the determination of independent restriction to each of them is a special problem. In the present paper we use the quark mass value, obtained in [3], as the input parameter and devote our attention to the gluon condensate.

2 Correlator of the axial-vector currents

We consider the correlator of the charmed quark axial-vector currents:

$$i \int d^4x e^{ipx} \langle T(j_\mu(x) j_\nu^+(0)) \rangle = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2) + q_\mu q_\nu \Pi_L(q^2). \quad (1)$$

Here $j_\mu = \bar{c} \gamma_\mu \gamma_5 c$, Π_L is the longitudinal part of the polarization operator. In what follows, only transverse part of the polarization operator $\Pi(q^2)$ is considered.

$\Pi(q^2)$ (1) can be expressed through its imaginary part with the help of dispersion relation:

$$\Pi(q^2) = \frac{q^2}{4\pi^2} \int_{4m^2}^{\infty} \frac{R(s) ds}{s(s - q^2)},$$

where

$$R(s) = 4\pi \text{Im} \Pi(s + i0), \quad (2)$$

m is the pole mass of c -quark.

The dispersion relation can be saturated by the contributions of physical states. In the axial-vector channel the only resonance with the mass $m_\chi = 3510.51 \pm 0.12 \text{ MeV}$ is known [6]. We use the simplest model of the spectrum, containing this resonance and continuum. As usual, in order to suppress the continuum contribution one should consider the derivatives of the polarization operator in Euclidean region $Q^2 = -q^2 > 0$:

$$M_n(Q^2) = \frac{4\pi^2}{n!} \left(-\frac{d}{dQ^2} \right)^n \Pi(Q^2) = \int_{4m^2}^{\infty} \frac{R(s) ds}{(s + Q^2)^{n+1}}. \quad (3)$$

Usually the quantities M_n are referred to as moments.

From the phenomenological point of view

$$M_n(Q^2) = 4\pi^2 \sum_{\chi} \frac{|\langle 0 | j_\mu | 0 \rangle|^2}{(m_\chi^2 + Q^2)^{n+1}},$$

and in the ratio of two successive moments the contribution of the lightest state dominates at large n :

$$\frac{M_{n-1}(Q^2)}{M_n(Q^2)} = m_\chi^2 + Q^2 + \delta_c,$$

where δ_c is the continuum contribution.

The QCD part of the polarization operator consists of the perturbative and nonperturbative terms:

$$\Pi(q^2) = \Pi^{\text{Pert}}(q^2) + \Pi^{\text{OPE}}(q^2).$$

The first one is determined by its imaginary part (2). $R(s)$ can be expressed as the series in the coupling constant α_s :

$$R(s) = \sum_{k=0,1,\dots} R^{(k)}(s, \mu^2) a^k(\mu^2). \quad (4)$$

Hereafter we denote $a(\mu^2) = \alpha_s(\mu^2)/\pi$. We will take first three terms in this series.

$R(s)$ is the physical quantity and does not depend on the scale μ^2 , but each term in

(4) can depend. First two terms of (4) are known analytically. They do not contain scale dependence. One can find them in [7]:

$$R^{(0)} = v^3, \quad (5)$$

$$R^{(1)} = \frac{4}{3} \left(v^2 \left((1+v^2)(2\ln(1-p)\ln p + \ln(1+p)\ln p + 2\text{Li}_2(p) + \text{Li}_2(p^2)) - \right. \right. \\ \left. \left. - 2v(2\ln(1-p) + \ln(1+p)) \right) - \frac{1-v}{32} (21 + 21v + 80v^2 - 16v^3 + 3v^4 + 3v^5) \ln p + \right. \\ \left. + \frac{3v}{16} (-7 + 10v^2 + v^4) \right). \quad (6)$$

In these expressions v is the quark velocity, $v = \sqrt{1 - 4m^2/s}$, $p = (1-v)/(1+v)$. Li_k is the polylogarithm function:

$$\text{Li}_k(v) = \sum_{n=1}^{\infty} \frac{v^n}{n^k}.$$

The term $R^{(2)}$ is represented usually as the sum of five gauge invariant parts:

$$R^{(2)} = C_F^2 R_A^{(2)} + C_A C_F R_{NA}^{(2)} + C_F T n_l R_l^{(2)} + C_F T R_F^{(2)} + C_F T R_S^{(2)}. \quad (7)$$

Here $C_A = 3$, $C_F = 4/3$, $T = 1/2$ are group constants and $n_l = 3$ is the number of light quarks.

The term $R_l^{(2)}$ corresponds to the so-called double-bubble diagram, the diagram with two quark loops, in external loop there are massive quarks, whereas in internal one – massless quarks. $R_l^{(2)}$ has the following form [7]:

$$R_l^{(2)} = \left(-\frac{1}{4} \ln \frac{\mu^2}{4s} - \frac{5}{12} \right) R^{(1)} + \delta_A^{(2)}, \quad (8)$$

where $\delta_A^{(2)}$ is given by equation (79) in [7].

$R_F^{(2)}$ appears from the double-bubble diagram with massive quarks in both loops. In the domain $s < 16m^2$ only the virtual massive quarks contribute, and $R_F^{(2)}$ has the form [7]:

$$R_F^{(2)} = 2v^3 \text{Re} F_{3,Q}^{(2)} - \frac{1}{4} \ln \frac{\mu^2}{m^2} R^{(1)}. \quad (9)$$

The expression for $F_{3,Q}^{(2)}$ can be found in appendix B of [7], equation(166).

For $s > 16m^2$ the contribution of the real quarks appears. It is given by double integral, which can not be taken analytically (equation (71) in [7]). The numerical calculation shows that it is small, nevertheless, we will take it into account.

The functions $R_A^{(2)}$ and $R_{NA}^{(2)}$ appear from diagrams with single quark loop and a number of gluon lines, they contain abelian and nonabelian exchanges correspondingly. These functions are not known analytically, consequently, one has to use some approximations.

Our approximation expressions are based on the first eight moments at $Q^2 = 0$, which are known analytically [8]. The obtained formulas, being substituted into (3), have to reproduce these eight moments with high accuracy. In order to construct such approximations we perform the following steps.

Let us consider the series

$$\Pi_{NA}^{(2)} = \frac{3}{16\pi^2} \sum_{k=1,2,\dots} C_{NA,k}^{(2)} z^k,$$

where $z = q^2/(4m^2)$, $\Pi^{Pert} = \sum_k \Pi^{(k)} a^k$ and

$\Pi^{(2)} = C_F^2 \Pi_A^{(2)} + C_A C_F \Pi_{NA}^{(2)} + C_F T n_l \Pi_l^{(2)} + C_F T \Pi_F^{(2)} + C_F T \Pi_S^{(2)}$ similar to (7). The coefficients $C_{NA,1}^{(2)}, \dots, C_{NA,8}^{(2)}$ are known analytically. First of all, we reexpand this series in terms of variable ω ,

$$\omega = \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}}.$$

Thus we map the complex q^2 -plane to the unit circle. Then we construct the Pade approximation, which usually gives a better accuracy than Taylor series. It has the following form:

$$\Pi(\omega) = \frac{a_0 + a_1\omega + \dots + a_i\omega^i}{1 + b_1\omega + \dots + b_j\omega^j}.$$

Since we have 8 moments in hand, we can construct Pade approximation with 8 parameters a_i and b_j . The best results turns out to give the following approximation:

$$\begin{aligned} \Pi_{NA}^{(2)}(\omega) &= \frac{3}{16\pi^2} \times \\ &\times \frac{10.8547\omega + 9.43221\omega^2 - 8.76722\omega^3 - 1.74256\omega^4 + 0.853743\omega^5 - 0.257734\omega^6}{1 + 0.373686\omega - 0.439076\omega^2}. \end{aligned} \quad (10)$$

In the similar way we obtain for the abelian part:

$$\begin{aligned} \Pi_A^{(2)}(\omega) &= \frac{3}{16\pi^2} \times \\ &\times \frac{9.25606\omega - 288.334\omega^2 - 394.513\omega^3 - 69.8439\omega^4 + 19.9673\omega^5 - 0.936404\omega^6}{1 - 32.0000\omega - 15.2803\omega^2}. \end{aligned} \quad (11)$$

The last term in (7) is generated by the diagram with two triangle quark loops (singlet part). We need the first moments in the expansion

$$\Pi_S^{(2)} = \frac{3}{16\pi^2} \sum_{k=1,2,\dots} C_{S,k}^{(2)} z^k. \quad (12)$$

Such diagrams for different currents were discussed in paper [9]. However, to cancel the axial anomaly, in the axial-vector case the current $\tilde{j}_\mu = \bar{c}\gamma_\mu\gamma_5 c - \bar{s}\gamma_\mu\gamma_5 s$ was considered.

As the result, in the calculated factors $\tilde{C}_{S,k}^{(2)}$ ($k = 1, \dots, 7$) the logarithms $\ln(q^2/m^2)$ appear due to massless cut (see equation (8) in [9]). For our purposes we need to subtract the contribution of the massless s -quark in $\tilde{C}_{S,k}^{(2)}$. The expression for the imaginary part $R_{Ss}^{(2)}$ (the contribution of the massless cuts) can be found in [10]. But the corresponding integral can not be taken analytically. The purely numerical integration is also impossible because of the presence of divergent logarithms $\ln(q^2/m^2)$ (which cancel out after subtracting from $\tilde{C}_{S,k}^{(2)}$). The suitable technique was suggested in [9]. $R_{Ss}^{(2)}$ is represented as the sum

$$R_{Ss}^{(2)} = \frac{9}{2} \ln \frac{s}{m^2} + R'_{Ss}{}^{(2)}.$$

Then the integral over $R'_{Ss}{}^{(2)}$ is divided into three parts:

$$\int_0^\infty \frac{R'_{Ss}{}^{(2)}(r) dr}{r-z} = \int_0^\epsilon \frac{R'_{Ss}{}^{(2)}(r) dr}{r-z} + \int_\epsilon^1 \frac{R'_{Ss}{}^{(2)}(r) dr}{r-z} + \int_1^\infty \frac{R'_{Ss}{}^{(2)}(r) dr}{r-z}. \quad (13)$$

In the first integral on the right hand side we expand $\tilde{R}'_{Ss}{}^{(2)}$ near $r = 0$ and take about 50 terms to obtain a stable sum of two first integrals in the range $\epsilon = 0.65 \dots 0.75$ with the high accuracy. The third term on the right hand side of (13) can be calculated purely numerically. Thus we obtain the coefficients in (12):

$$\begin{aligned} C_{S,1}^{(2)} &= -0.20665154, \\ C_{S,2}^{(2)} &= -0.063212891, \\ C_{S,3}^{(2)} &= -0.024202688, \\ C_{S,4}^{(2)} &= -0.01062245, \\ C_{S,5}^{(2)} &= -0.0050245436, \\ C_{S,6}^{(2)} &= -0.002441955, \\ C_{S,7}^{(2)} &= -0.0011555635. \end{aligned}$$

On the basis of these numbers one can obtain the following approximation:

$$\begin{aligned} \Pi_S^{(2)}(\omega) &= \frac{3}{16\pi^2} \times \\ &\times \frac{-0.826606\omega + 0.179208\omega^2 + 0.5387\omega^3 - 0.350049\omega^4 + 0.0425326\omega^5}{1 + 0.559635\omega - 0.196815\omega^2}. \quad (14) \end{aligned}$$

It should be noted that the purely gluonic cut in this diagram is zero according to the Landau-Yang theorem [11].

In the last step we take the imaginary part $R_\delta^{(2)}$, $\delta = NA, A, S$ of the (10),(11),(14) ($\mu^2 = m^2$):

$$\begin{aligned} R_\delta^{(2)} &= 4\pi \text{Im} \Pi_\delta^{(2)}(\omega), \\ \omega &= \frac{1 + i\sqrt{z-1}}{1 - i\sqrt{z-1}}. \end{aligned} \quad (15)$$

The nonperturbative part of the polarization operator was calculated up to the operators of dimension 8 in [12]. For the correlator of heavy quarks there are: the only operator of dimension 4,

$$O_2 = \langle g^2 G_{\mu\nu}^a G_{\mu\nu}^a \rangle,$$

two operators of dimension 6,

$$O_3^1 = \langle g^3 f^{abc} G_{\mu\nu}^a G_{\nu\eta}^b G_{\eta\mu}^c \rangle, \quad O_3^2 = \langle g^4 j_\mu^a j_\mu^a \rangle,$$

and seven operators of dimension 8:

$$\begin{aligned} O_4^1 &= \langle (g^2 d^{abc} G_{\mu\nu}^b G_{\eta\lambda}^c)^2 + \frac{2}{3} (g^2 G_{\mu\nu}^a G_{\eta\lambda}^a)^2 \rangle, \quad O_4^2 = \langle (g^2 f^{abc} G_{\mu\nu}^b G_{\eta\lambda}^c)^2 \rangle, \\ O_4^3 &= \langle (g^2 d^{abc} G_{\mu\eta}^b G_{\eta\nu}^c)^2 + \frac{2}{3} (g^2 G_{\mu\eta}^a G_{\eta\nu}^a)^2 \rangle, \quad O_4^4 = \langle (g^2 f^{abc} G_{\mu\eta}^b G_{\eta\nu}^c)^2 \rangle, \\ O_4^5 &= \langle (g^5 f^{abc} G_{\mu\nu}^a j_\mu^b j_\nu^c) \rangle, \quad O_4^6 = \langle (g^3 f^{abc} G_{\mu\nu}^a G_{\nu\eta}^b G_{\eta\mu;\lambda\lambda}^c) \rangle, \quad O_4^7 = \langle g^4 j_\mu^a j_{\mu;\eta\eta}^a \rangle. \end{aligned}$$

Here j_μ^a is the color current of the light quarks, $g j_\mu^a = G_{\mu\nu;\nu}^a = \frac{g}{2} \sum_{q=u,d,s} \bar{q} \gamma_\mu \lambda^a q$. The operator product expansion part Π^{OPE} of the polarization operator has the form [12] ($y = Q^2/(4m^2)$, ${}_2F_1(a, b, c, z)$ is hypergeometric function):

$$\begin{aligned} \Pi^{OPE}(y) &= \frac{1}{4\pi^2} \frac{O_2}{(4m^2)^2} \left(\frac{1}{3y} - \frac{2}{5} {}_2F_1(1, 4, 7/2, -y) \right) + \\ &+ \frac{1}{27\pi^2} \sum_{j=1}^2 \frac{O_3^j}{(4m^2)^3} \left(\frac{c_{3,0}^j}{y} + \sum_{i=1}^5 c_{3,i}^j {}_2F_1(1, 2+i, 9/2, -y) \right) + \\ &+ \frac{1}{27\pi^2} \sum_{j=1}^7 \frac{O_4^j}{(4m^2)^4} \left(\frac{c_{4,0}^j}{y} + \sum_{i=1}^7 c_{4,i}^j {}_2F_1(1, 3+i, 11/2, -y) \right). \end{aligned} \quad (16)$$

In this equation

$$\begin{aligned} c_{3,0}^1 &= -3/5, \quad c_{3,1}^1 = 0, \quad c_{3,2}^1 = 0, \quad c_{3,3}^1 = 24/7, \quad c_{3,4}^1 = -116/7, \quad c_{3,5}^1 = 552/35, \\ c_{3,0}^2 &= 36/5, \quad c_{3,1}^2 = 8/105, \quad c_{3,2}^2 = 8/35, \quad c_{3,3}^2 = 144/35, \quad c_{3,4}^2 = 32/21, \quad c_{3,5}^2 = -736/35, \\ c_{4,i}^j &= \end{aligned}$$

$j \backslash i$	0	1	2	3	4	5	6	7
1	0	2/315	-16/63	92/21	544/63	-562/9	160/3	0
2	9/35	1/105	32/63	-118/9	200/3	-5639/45	30448/315	-180/7
3	-18/5	-8/315	64/63	-752/21	9920/63	-10036/45	320/3	0
4	-144/35	-2/315	-32/45	-188/63	7376/63	-10586/45	-26464/315	1656/7
5	612/35	-4/105	-64/63	184/7	-4000/21	20524/45	-1984/7	-432/7
6	72/35	0	-128/315	944/63	-432/7	-1544/45	98944/315	-1728/7
7	-324/35	4/315	0	40/9	-1696/63	292/3	-14528/63	1296/7

The α_s -correction of the first order to the gluon condensate was obtained in [13]. The corresponding term in the polarization operator has the following form:

$$\Pi^{O2(1)}(Q^2) = \frac{1}{4\pi^2} \frac{O_2}{(4m^2)^2} a(m^2) \frac{P^A(-y)}{8y^2(1+y)}, \quad (17)$$

where $P^A(z)$ is given in equation (8) of [13].

3 Moments and mass redefinition

In our approach the moments contain the perturbative part up to α_s^2 terms, the operator product expansion part with the operators of 4, 6 and 8 dimensions and perturbative correction of the first order to the lowest dimension operator:

$$M_n(Q^2) = \sum_{k=0}^2 M_n^{(k)}(Q^2) a^k(m^2) + M^{OPE} + O_2 M_n^{O2(1)}(Q^2) a(m^2), \quad (18)$$

$$M^{OPE} = O_2 M_n^{O2}(Q^2) + \sum_{j=1}^2 O_3^j M_n^{O3,j}(Q^2) + \sum_{j=1}^7 O_4^j M_n^{O4,j}(Q^2).$$

The explicit formula for $M_n^{(0)}$ can be obtained analytically:

$$M_n^{(0)} = \frac{1}{(4m^2)^n} \frac{3\sqrt{\pi}(n-1)!}{4\Gamma(n+\frac{5}{2})} {}_2F_1(n, 1+n, 5/2+n, -y),$$

where $y = Q^2/(4m^2)$. The next perturbative terms in (18) we calculate numerically (see (3)):

$$M_n^{(k)}(Q^2) = \frac{1}{(4m^2)^n} \int_1^\infty \frac{R^{(k)}(r, m^2) dr}{(r+y)^{n+1}}.$$

Here $R^{(1)}$ is given by (6), for the components of $R^{(2)}$ see (7), (8), (9) and (15), (10), (11), (14). The operator product expansion moments can be easily obtained from (16):

$$\begin{aligned} M_n^{O2} &= \frac{1}{(4m^2)^{n+2}} \left(\frac{1}{3y^{n+1}} - \frac{(3+n)! \Gamma(7/2)}{15 \Gamma(7/2+n)} {}_2F_1(1+n, 4+n, 7/2+n, -y) \right), \\ M_n^{O3,j} &= \frac{4}{27(4m^2)^{n+3}} \left(\frac{c_{3,0}^j}{y^{n+1}} + \sum_{i=1}^5 c_{3,i}^j \frac{(1+i+n)! \Gamma(9/2)}{(1+i)! \Gamma(9/2+n)} {}_2F_1(1+n, 2+i+n, 9/2+n, -y) \right), \\ M_n^{O4,j} &= \frac{4}{27(4m^2)^{n+4}} \left(\frac{c_{4,0}^j}{y^{n+1}} + \sum_{i=1}^7 c_{4,i}^j \frac{(2+i+n)! \Gamma(11/2)}{(2+i)! \Gamma(11/2+n)} {}_2F_1(1+n, 3+i+n, 11/2+n, -y) \right). \end{aligned}$$

As for α_s -correction to the operator O_2 , the corresponding moments can be obtained by numerical differentiation of (17):

$$M_n^{O2(1)} = \frac{1}{(4m^2)^2 n!} \left(-\frac{d}{dQ^2} \right)^n \frac{P^A(y)}{8y^2(1+y)},$$

$$y = Q^2/(4m^2).$$

Calculating the numerical values of the moments one can easily find that the α_s -corrections to the moments are very large and in fact the series (18) is divergent. The traditional solution of this problem is the mass redefinition. Indeed, the pole mass in the above formulas has the meaning of the mass of free quark and is ill defined quantity in the charmonium sum rules. This problem was discussed in details in [3]. Following [3], we consider $\overline{\text{MS}}$ scheme. The corresponding mass \bar{m} is taken at the scale $\mu^2 = \bar{m}^2$: $\bar{m} = \bar{m}(\bar{m}^2)$.

The pole mass m can be expressed in terms of \bar{m} as the perturbative series:

$$m^2 = \bar{m}^2 \left(1 + \sum_{k=1,2,\dots} K_n a^k(\bar{m}) \right),$$

where K_1, K_2, K_3 were obtained in [14]:

$$K_1 = 8/3, \quad K_2 = 22.4162, \quad K_3 = 260.526.$$

(These numbers are given for $n_l = 3$.)

Now we reexpand the series (18) in terms of the mass \bar{m} :

$$M_n(Q^2) = \sum_{k=0}^2 \bar{M}_n^{(k)}(Q^2) a^k(m^2) + \bar{M}^{OPE} + O_2 \bar{M}_n^{O2(1)}(Q^2) a(m^2). \quad (19)$$

Here

$$\bar{M}_n^{(0)}(Q^2) = M_n^{(0)},$$

$$\bar{M}_n^{(1)}(Q^2) = M_n^{(1)} - K_1 n M_n^{(0)} + K_1(n+1) Q^2 M_{n+1}^{(0)},$$

$$\begin{aligned} \bar{M}_n^{(2)}(Q^2) = & M_n^{(2)} - K_1 n M_n^{(1)} + K_1(n+1) Q^2 M_{n+1}^{(1)} + n \left(\frac{K_1^2}{2} (n+1) - K_2 \right) M_n^{(0)} + \\ & + (n+1) \left(K_2 - K_1^2 (n+1) \right) Q^2 M_{n+1}^{(0)} + \frac{K_1^2}{2} (n+1)(n+2) Q^4 M_{n+2}^{(0)}, \end{aligned}$$

$$\bar{M}_n^{O2}(Q^2) = M_n^{O2}, \quad \bar{M}_n^{O_{k,j}}(Q^2) = M_n^{O_{k,j}}, \quad k = 3, 4, \quad \bar{M}_n^{OPE}(Q^2) = M_n^{OPE},$$

$$\bar{M}_n^{O2(1)}(Q^2) = M_n^{O2(1)} - K_1(n+2) M_n^{O2} + K_1(n+1) Q^2 M_{n+1}^{O2}.$$

All moments in the right hand sides of these equations are computed with the $\overline{\text{MS}}$ mass \bar{m} .

At the some other scale μ^2 the function $\bar{M}_n^{(2)}$ changes

$$\bar{M}_n^{(2)}(Q^2) \rightarrow \bar{M}_n^{(2)}(Q^2) + \bar{M}_n^{(1)}(Q^2) \beta_0 \ln \frac{\mu^2}{\bar{m}^2}, \quad a(\bar{m}^2) \rightarrow a(\mu^2) \quad (20)$$

to ensure the scale independence at order α_s^2 .

4 Restrictions on the gluon condensate value

To analyse the obtained data we introduce the dimensionless ratio r_n of the moments (19):

$$r_n = \frac{M_{n-1}(Q^2)}{4\bar{m}^2 M_n(Q^2)} = \frac{m_\chi^2 + Q^2}{4\bar{m}^2} + \delta \quad (21).$$

Here δ stands for the continuum contribution. At large n δ tends to zero: $n \rightarrow \infty$, $\delta \rightarrow 0$. The theoretical ratio of the moments depends on Q^2 , quark mass \bar{m} , QCD coupling and condensates. First of all, let us fix Q^2 . At $Q^2 = 0$ the perturbative corrections as well as the higher terms of the operator product expansion series are very large. On the other hand, at large Q^2 , $Q^2/(4\bar{m}^2) \geq 3$, the effective expansion parameter $a\beta_0 \ln(Q^2/\bar{m}^2)$ in (20) becomes large. In papers [3],[5] the values $Q^2/(4\bar{m}^2) = 1$, $Q^2/(4\bar{m}^2) = 2$ were used. In the present paper we work at $Q^2/(4\bar{m}^2) = 2$.

As for the QCD coupling constant, from the hadronic τ -decay we know [15]:

$$\alpha_s(m_\tau^2) = 0.33 \pm 0.03, \quad (22)$$

$m_\tau = 1.777 \text{ GeV}$ is the mass of the τ -lepton. To obtain α_s at any other scale we solve numerically the renormalization group equation. The choice of the scale is discussed in [3]:

$$\mu^2 = Q^2 + \bar{m}^2. \quad (23)$$

The value of the c -quark mass \bar{m} is determined in [3],[5] with high accuracy:

$$\bar{m} = 1.275 \pm 0.015 \text{ GeV}$$

(this value is taken from [3]). Now theoretical r_n depends on the condensates only.

Using the value $m_\chi = 3.51051 \pm 0.00012 \text{ GeV}$ [6] we find the phenomenological estimation of r_n in (21):

$$r_n = \frac{m_\chi^2 + Q^2}{4\bar{m}^2} + \delta = 3.90 \pm 0.05 + \delta. \quad (24)$$

The uncertainty appears mainly due to the error in the quark mass value.

Now let us look on the table of the moments.

n	$a\bar{M}^{(1)}/\bar{M}^{(0)}$	$a^2\bar{M}^{(2)}/\bar{M}^{(0)}$	$\bar{M}^{OPE}/\bar{M}^{(0)}$	$a\bar{M}^{O2(1)}/\bar{M}^{O2}$
4	0.097	0.032	0.083	-0.83
5	0.070	0.018	0.20	-0.51
6	0.041	0.0013	0.44	-0.27
7	0.0087	-0.017	0.91	-0.065
8	-0.026	-0.037	1.80	0.12
9	-0.061	-0.058	3.47	0.30

In this table $Q^2/(4\bar{m}^2) = 2$, $\langle \frac{\alpha_s}{\pi} G^2 \rangle / (4\bar{m}^2) = 0.0002$ and scale (23) are chosen.

Operators O_3^k , O_4^k are excluded.

The values in all the columns in the certain line should be small enough (< 0.5 in magnitude), besides, the values in the third column should be smaller than in the second

one. All these requirements ensure the convergence of the series (19). Thus we see that there is very narrow range of n : $n = 5 - 6$, and one can construct the only ratio (21):

$$r_6 = \frac{M_5}{4\bar{m}^2 M_6}.$$

For $Q^2/(4\bar{m}^2) = 1$ there are no appropriate n at all.

The values of the higher order operators, O_3^k, O_4^k , can be estimated with the help of factorization hypothesis or instanton model (for details see [5]). In the instanton consideration

$$O_3^1 = \frac{12}{5\rho_c^2} O_2, \quad O_3^2 = 0$$

$$(O_4^1, \dots, O_4^7) = (4, 8, 3, 4, 0, 8, 0) \frac{16}{7\rho_c^4} O_2.$$

Here instanton radius $\rho_c = 2.5 \text{ GeV}^{-1}$. The instanton concentration n_0 is connected with the gluon condensate: $\langle \frac{\alpha_s}{\pi} G^2 \rangle = 32n_0$.

In the frame of the factorization hypothesis

$$(O_4^1, \dots, O_4^7) = \left(\frac{65}{144}, \frac{5}{16}, \frac{19}{72}, \frac{1}{16}, 0, \frac{1}{8}, 0 \right) (O_2)^2.$$

However, at $n = 5 - 6$ both models give negligible corrections.

At first we put $\delta = 0$ in (24). Then $3.85 \leq r_6 \leq 3.95$. Now we find the interval of the gluon condensate values, where these restrictions hold:

$$0.005 \text{ GeV}^4 \leq \langle \frac{\alpha_s}{\pi} G^2 \rangle \leq 0.006 \text{ GeV}^4.$$

The uncertainty in the coupling constant and variation of the scale ($\mu^2 = Q^2$ or $\mu^2 = Q^2 + 2\bar{m}^2$) give negligible correction to the value of gluon condensate.

Now let us try to evaluate δ . Unlike to the vector channel, the mass of the second resonance with $J^{PC} = 1^{++}$ is unknown. If we suppose that the differences between two lowest resonances in vector and axial-vector channels are approximately equal (about 0.6 GeV), the continuum threshold can be evaluated: $s_0 \approx 4.1 \text{ GeV}$. Now one can compare the integrals like (3) in the intervals $[4\bar{m}^2, \infty)$ and $[s_0^2, \infty)$ to evaluate the continuum contribution to the certain moment. In such way we find:

$$\delta < 0.2.$$

It is especially important that δ is positive, $\delta > 0$.

It turns out that in the considered sum rules the increase of r_6 results in decrease of the value $\langle \frac{\alpha_s}{\pi} G^2 \rangle$ (see fig.1). That is why introducing the positive δ in (24) we obtain the smaller values of the gluon condensate.

In some papers in the sum rules for heavy quarks the Coulomb-like corrections are summed up. This is legitimate way for the nonrelativistic problems only. However, in our case the quark velocities $v \approx \sqrt{(1 + Q^2/(4\bar{m}^2))/n} \approx 0.7$ are large enough, and the nonrelativistic corrections are not dominating. Detailed consideration of this question can

be found in [3].

Our final result is:

$$\langle \frac{\alpha_s}{\pi} G^2 \rangle = (0.005 + 0.001 - 0.004) \text{ GeV}^4. \quad (25)$$

Note again that the continuum (i.e. higher states) does not affect the upper limit in (25), which can be considered as quite reliable. As for the lower limit, it depends on δ , for which one has the estimation only. Therefore, even zero value of the condensate can not be certainly excluded.

Our result (25) is in agreement with the values of the gluon condensate, obtained in [3] ($\langle \frac{\alpha_s}{\pi} G^2 \rangle = 0.009 \pm 0.007 \text{ GeV}^4$) and in [5] ($\langle \frac{\alpha_s}{\pi} G^2 \rangle < 0.008 \text{ GeV}^4$). However, it is significantly smaller than 0.012 GeV^4 , originally obtained in [1].

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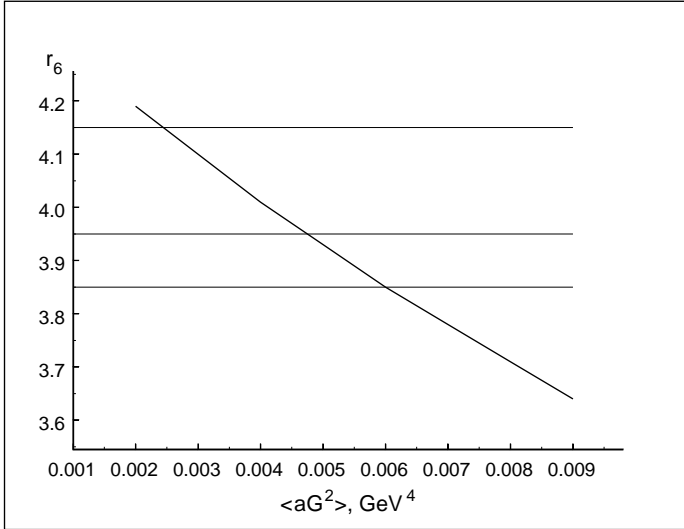


Figure 1: The ratio r_6 as the function of the gluon condensate. Horizontal lines denote the limits in (24).

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